THE EFFECTIVENESS OF VARIATION AS PART OF THE
MASTERY APPROACH TO TEACHING PRIMARY
MATHEMATICS

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Abstract

This literature based project centers around variation, as part of the mastery approach, and its benefits to achieving mathematically fluent pupils. The primary aim of the research is to build personal knowledge to further improve mathematical pedagogy. The issues and debates explored in the literature review highlight the need for a change in mainstream primary mathematics teaching, and discusses what successful educating systems are currently doing.

The project was conducted using qualitative secondary research. Through this research, the project examines the definition of mathematical variation, explores what makes variation successful, and analyses examples of variation. This project concludes that successful variation results in pupils who have deep conceptual and procedural knowledge, and can therefore become mathematically fluent.

Some implications of this research are, that teachers need time and support to collaborate and develop professionally in order to execute effective variation. And when used as part of a mastery approach, variation assists in achieving aims such as, high expectations of all pupils, progression through content at the same pace, and understanding and fluency in mathematical concepts is successfully built.
# Contents

Abstract ....................................................................................................................... iv  
Chapter 1 – Introduction......................................................................................... 1  
Chapter 2 – Literature Review ............................................................................. 4 
  2.1 What is Effective Primary Mathematics Teaching? ......................................... 4 
  2.2 What is a Mastery Curriculum? ....................................................................... 7  
  2.3 Why Does England not Follow a Mastery Approach? ....................................... 8  
Chapter 3 – Methodology ...................................................................................... 11 
Chapter 4 – Defining Variation ............................................................................. 13 
  4.1 What is Variation? ............................................................................................ 13  
  4.2 Procedural and Conceptual Variation ................................................................. 14 
  4.3 How Variations fits in the English Curriculum .................................................. 16 
Chapter 5 – What Makes Variation an Effective Teaching Approach? ................. 18 
  5.1 Depth Before Breadth ....................................................................................... 18 
  5.2 Quality of Teaching ......................................................................................... 19 
  5.3 High Expectations of all Pupils ........................................................................ 20 
Chapter 6 – Analysing Examples of Variation ....................................................... 22 
  6.1 Example 1 (Appendix 2) .................................................................................. 22 
  6.2 Example 2 (Singapore, Maths Publishers, 2015, p.3 & p.43) ......................... 23 
  6.3 Example 3 (Foong et al., 2015, p.60) ............................................................... 24 
  6.4 Example 4 (Gu et al., 2004, p.316) ................................................................. 26 
  6.5 Example 5 (Watson and Mason, 2006, p.104) .................................................. 28 
Chapter 7 – Conclusion ......................................................................................... 29 
References .............................................................................................................. 32 
Appendices ............................................................................................................ 37 
Appendix 1 – Meeting Log ..................................................................................... 37 
Appendix 2 – Calculation Guidance for Primary Schools ..................................... 40
Chapter 1 – Introduction

The new national curriculum (2013, p.3) has placed fluency, problem solving and reasoning at the centre of the primary mathematics agenda. In doing so, the Department for Education has attempted to reflect the curriculums found in internationally high performing countries, where on average, pupils are leaving school academically three years ahead in mathematics than pupils in England (NCETM, 2014). Consequently, Ofsted and individual schools are now paying more attention to approaches, such as mastery, to develop these critical skills in pupils.

The mastery approach is common to East Asian education systems such as Singapore and Shanghai, and the effectiveness of the approach has led to much research being conducted to help improve other educational systems (Drury, 2014; Ginsburg et al., 2005). However, although it is appreciated that the approach is effective, very few schools in England have implemented a mastery approach to teaching mathematics. Fluency, opportunities for problem solving and reasoning mathematically are all new aims in the English national curriculum (Department for Education, 2013, p.3), however research has emphasised the qualities and necessities of these aspects previously (Cockcroft, 1982; x; Barmby et al., 2007; Haylock, 2010). The previous national curriculum (Department for Education and Employment, 1999, p.62), briefly mentioned that ‘pupils develop their understanding and knowledge of mathematics through practical activity, exploration and discussion’. This was stated on the left hand side of page 62 of teaching requirements, and in grey font, which could easily be dismissed. The new national curriculum (2013, p.3) however, places fluency, problem solving and reasoning as focal aims, boldly at the beginning of the mathematics programme of study, hence placing significant importance to these terms.

The mastery approach can be characterised by the following features:

- High expectations that every pupil can achieve in mathematics (NCETM, 2014; Drury, 2014, p.8).
The majority of pupils progress through the curriculum at the same pace, achieving differentiation through depth of understanding (Drury, 2014, p.27; Stripp, 2014).

Precise vocabulary is used in questioning and discussion to develop mathematical justifications and reasoning in pupils. And questioning is regularly used to assess pupils, and identify those in need of intervention (NCETM, 2014; Mathematics Mastery, 2015).

Carefully chosen models, images and resources are used to foster deep understanding, which underpins teaching. Whilst meticulous variation is structured through practice and consolidation to build fluency (NCETM, 2014; Haylock, 2010, p.18).

The characteristics of the mastery approach enable the three key aims of the new national curriculum to be achieved in every pupil. As access to the full curriculum for all, encourages confidence and competence in all areas of mathematics, hence equipping pupils with the mathematical skills needed for their futures.

The research project was chosen, as I have a particular interest in primary mathematics. Additionally, society’s acceptance at people being bad at mathematics, and teacher’s willingness to admit they lack confidence in mathematics, inspired me to further develop my understanding of what effective mathematics teaching is and how it can be implemented in primary classrooms. The purpose of this study is to evaluate variation, as an element of the mastery approach, for its effectiveness in achieving conceptual and procedural fluency in pupils. It will also attempt to assess what makes variation successful in countries where it is currently in use, and how it can be implemented into English primary mathematics classrooms. These aims will be achieved through developing understanding of the concept of variation, how variation meets the needs of the national curriculum, and analysing examples of variation. This study will be divided into chapters, which will analyse and synthesise qualitative secondary
research, hence forming the basis of this project. The final chapter will conclude with an evaluation of the findings, how the study can be implemented to meet national curriculum requirements, and make recommendations for future research around the topic.
Chapter 2 – Literature Review

Mathematics is a core subject in the national curriculum, which has the capability to engage a learner with the emotional satisfaction of making sense of numbers, patterns and shapes. Yet, for far too many children mathematics has become a set of unconnected, meaningless rules which when used correctly lead to a correct answer (Haylock, 2010, p.15; Skemp, 1971, p.1). This chapter will explore: what makes effective primary mathematics teaching; what the mastery approach to primary mathematics teaching is; and discuss the arguments behind the need for England to change its approach to mathematics teaching.

2.1 What is Effective Primary Mathematics Teaching?

Developing depth of understanding is broadly agreed as essential when teaching mathematics (Haylock, 2010, p.3; Department for Education, 2013, p.3; Barmby et al., 2010, p.47). Pupils, for whom the curriculum focuses around deep understanding of the content, have the ability to make connections within mathematics and to the wider world, solve problems efficiently, and therefore become fluent in mathematical application and language (Haylock, 2010, p.25; Barmby et al., 2010, p.52; Barmby et al., 2007, p.42). When this is achieved, rather than regurgitating number facts, learning becomes meaningful, which results in pupils being able to explore and engage in a world where mathematics features daily (Cockcroft, 1982, p.1; Bobis, 2007, p.26). Furthermore, pupils need to be exposed to rich problem solving tasks, which are structured specifically to help them make connections whilst challenging all abilities (Drury, 2014, p.24; Department for Education, 2013, p.3). In order to create such tasks, teachers need to have confidence and profound subject knowledge (Haylock, 2010, p.3; Department for Children, Schools and Families, 2008, p.6). These requirements create effective teachers who can ensure pupils develop good habits when reasoning, looking for patterns, and making connections in mathematics. Teachers who have strong mathematical subject knowledge also have the ability to immerse pupils in precise and appropriate mathematical vocabulary (Cockcroft, 1982, p.90; Hansen, 2008, p.7). By
doing so, mathematical discussion is ignited and plays an essential role when developing understanding.

Depth of understanding, problem solving, making connections, using mathematical vocabulary, and teachers with strong subject knowledge, are all elements which Singapore have mastered effectively, and are factors to why they are ranked first internationally for mathematics education (TIMSS, 2011). Figure 1 is taken from the Singapore Mathematics Syllabus (MoE, 2013, p.14), which emphasises their commitment to teaching through problem solving to develop understanding. Additionally, figure 2 (MoE, 2013, p.16) highlights how teachers should ensure their pupils are making connections through problem solving, with opportunities to reflect and take ownership of their own learning. Both, figure 1 and 2, come from Singapore’s four page chapter about problem solving, which highlights their commitment, and the importance they place, on teaching and learning through problem solving. The Ministry of Education for Singapore (2003, p.8) comment on their success, and the need for continual success, of every pupil through encouragement, stimulation and assessment. Their commitment to having high expectations for every pupil is vindication for the fact that on average by age 15 pupils in Singapore are, mathematically,
three academic years above pupils of the same age in England (PISA, 2012). Singapore, along with many other high achieving East Asian countries, follow a mastery curriculum which incorporates these features to produce an outstanding mathematics education. The importance of developing these mathematical skills in all pupils, combined with the knowledge of an approach, which is currently enabling countries to achieve these skills in their pupils successfully, provides a rationale for analysing the features of a mastery curriculum.

![Diagram of the Mathematical Modelling Process](image-url)
2.2 What is a Mastery Curriculum?

The mastery approach follows a set of principles, which focus on covering depth before breadth (Drury, 2014, p.24; Mathematics Mastery, 2015). In addition to East Asian countries, Western countries such as Norway and Finland also follow a mastery approach. The approach enables teachers to ensure pupils have time to master key concepts and ideas, through rich problem solving, as opposed to attempting to cover a vast number of concepts. These higher achieving countries also insist on all pupils reaching the same high standards, by ensuring they study the same content (Drury, 2014, p.27; NCETM, 2014). By having high expectations of all pupils, everyone is given the same access to content. The challenge for more able pupils is provided through increased depth of the same content, rather than acceleration. And regular formative assessment provides teachers with the knowledge of when, and who, to give individual support or intervention, hence ensuring they keep up with the rest of the class (Ginsburg et al., 2005, p.8; Drury, 2014, p.27).

Variation is another key principle in mastery. All teaching is supported by carefully chosen resources which foster deep understanding, both conceptually and procedurally, for pupils whilst enabling them to be actively involved in their learning (Gu et al., 2004, p.309; NCETM, 2014). By exploring concepts, both concretely and abstractly, pupils can begin to contextualise the mathematics, developing mathematical justifications and reasoning which results in pupils mastering mathematics (Gu et al., 2004, p.315; Appendix 2). Through variation, the teacher is also able to provide a multitude of questions and/or resources, which guide pupils to actively construct, develop and find connections between concepts (Lai and Murray, 2012, p.6; Watson and Mason, 2006, p.92). Additionally, variation enables pupils to focus on finding the relationships and patterns between numbers and concepts rather than teaching procedures without meaning or understanding (Appendix 3). The well designed sequence of variation tasks and resources, invite pupils to reflect on their learning and use higher order thinking skills to develop their understanding (Lai and Murray, 2012, p.6; Gu et al., 2004, p.315; Simon and Tzur, 2004, p.94).
In addition, teachers themselves are a key principle in the mastery approach. In some countries all the primary mathematics teachers are specialists in the subject. However, in all countries which follow a mastery approach, the emphasis on deep subject knowledge, time to plan collaboratively and opportunities for teachers to continually improve, are fundamental to the success (NCETM, 2014; Ginsburg et al., 2005, p.8; Mathematics Mastery, 2015). As a result of this: teachers have a strong understanding of the structure and aims of the curriculum; the ability to select, share and employ effective mathematical representations; and the capability to develop and use precise questioning to test conceptual and procedural development, as well as identify those in need of intervention (NCETM, 2014; Lemov, 2010, p.47). This brief outline of the mastery approach, and its dominance in international mathematics rankings provides a rationale for exploring why it is not currently implemented in English primary classrooms.

2.3 Why Does England not Follow a Mastery Approach?

In the 1982 Mathematics Counts report, Cockcroft (1982, p.96) emphasised the importance of increasing the opportunities for pupils to use and apply mathematics in the classroom. However, since then little change has been seen, with many continuing to argue the need for an approach which helps develop understanding for all (Ofsted, 2008, p.7; Haylock, 2010, p.24; Drury, 2014, p.22). Limited problem solving occurring in classrooms, and little evidence of pupils being equipped with the mathematics they need for their future, are comments made by Ofsted in the Mathematics: Understanding the Score report (2008, p.7). Yet, Barmby et al. (2007, p.45) discusses the potential for change, and the need for an approach, such as mastery, to increase the amount of mathematically rich problem solving tasks in every day mathematics lessons. Through the mastery approach, pupils are exposed to rich problem solving which develops mathematical thinking and reasoning, as opposed to expecting pupils to remember procedures with no connection or meaning (Barmby et al., 2010,
Changes to the national curriculum (2013, p.3), have increased flexibility for teachers, the previous structure however, may have been a barrier, which has now been lifted to potentially make way for developing understanding in all pupils (Drury, 2014, p.20). However, Houssart (2004, p.166) argued that a mastery approach is problematic, and that it is an ideal to imply that every pupil can master a particular concept. Additionally, Ginsburg (2005, p.x) contrasted that a mastery approach would be effective, yet the new national curriculum (2013) was the issue. As unlike successful mathematics educational systems, the flexibility removes national standards and prevents England ensuring an effective and consistent mathematics curriculum for all.

Ginsburg (2005, p.xi) continues to debate that the increase in academies prevents a mastery approach, as it enables unqualified and untrained people to become teachers. Considerations of this kind are supported by Haylock (2010, p.3), who believes the best way for pupils to learn is from teachers who understand, and have mastered, the subject themselves. In concurrence, teachers who had strong subject knowledge, were able to guide their pupils to make connections within mathematics and to the wider world, and these pupils saw far greater progress than those who were unable to make connections (Askew, 1997, p.64; Ofsted, 2008, p.7, Department for Children, Schools and Families, 2009, p.8). The Department for Children, Schools and Families (2008, p.8) agreed, stating that the most effective pedagogical approach was good quality mathematics subject knowledge. In China teachers receive 10-12 years of training, and in Singapore teachers are granted 100 hours of professional training per annum (Gu et al., 2004, p.310; Wong, 1999). In contrast, English teachers can have as little as 1 year of formal teacher training, and there is currently no minimum requirement for further professional development throughout a teacher’s career. Hence, a larger number of teachers in China and Singapore have greater confidence and ability to develop rich problems, appropriate question variations, and precise verbal questioning to deepen the understanding of their pupils.
Ofsted (2008, p.6) also found imbalances in the quality of teaching, which affected the opportunities different ages, abilities and special needs, were receiving across schools. Therefore, it is essential that both teachers and schools adopt an ethos that promotes deep understanding and progression for all. By employing a mastery approach, schools can reduce the barriers that prevent some pupils from achieving. Often low attaining pupils received restricted curriculum content, forcing them to become further behind, whilst high-achieving pupils were set poor extension tasks, which can foster the idea that success in mathematics is like a race (Stripp, 2014). On reflection: opportunities for pupils to use and apply mathematics; quality of teachers subject knowledge; and restrictive differentiation, currently prevents English primary classrooms from having an entirely effective mathematics curriculum. However, implementing a mastery approach, could resolve these barriers.

Variation theorists such as Anne Watson, John Mason and Gu Lingyuan, discuss the ability of structured, focused and connective mathematical tasks to develop conceptual understanding, fluency, accuracy and interest in pupils (Watson and Mason, 2006, p.97; Gu, 2004, p.315). And when variation is applied to as many mathematics concepts and lessons as possible, it shifts the pupil’s ability from using a concept, to perceiving relationships within and between concepts, and making them applicable to other situations (Mason, cited in: Watson and Mason, 2004, p.92). Therefore, this research project will aim to analyse and evaluate the effectiveness of variation, as a key element of the mastery approach, by discussing the below:

1. What is the definition of variation?

2. What makes variation an effective teaching approach?

3. Analysing examples of variation.
Chapter 3 – Methodology

This chapter will address the nature of the research methods employed and justify the reasoning behind the decisions made. The purpose of this research project is to investigate what makes effective primary mathematics teaching, and through doing so, examine the mastery approach with specific focus on variation. It will conclude with a concise summary of the findings and the professional impact the research has had on myself.

As mastery, and in turn variation, are long term approaches to enhance long term progression, the conclusion was made that conducting primary research would not produce accurate results that were a true reflection of the impact of the approaches. While it would have been possible to teach lessons in a mastery style, and then interview or issue questionnaires to teachers and pupils, the answers would have been subjective, due to the approach having not been followed for a long enough period. As pupils will not have been fully immersed into the mastery approach, including variation, it can appear as repetitive and a slower pace at first, however when followed long term, develops deeper understanding and fluency.

Consequently, it is therefore more suitable to conduct secondary research in the form of an extended literature review. This involves the analysis, synthesis and evaluation of existing research to form new ideas and interpretations, and similarly to how primary research projects are formed, make conclusions from the data gathered (Paniagua, 2002, p.22; Merriam, 1988, p.178).

From using existing literature, a narrower investigation into the effectiveness of variation, as part of mastery, was developed. Variation’s current role in primary mathematics, the effects it has on deepening understanding and therefore progressing pupils, and the different types of variation became central research elements of this dissertation. To do so, literature published from countries currently using variation, and literature
published to support primary mathematics teachers in England, were used to analyse the effectiveness.

In order to locate the literature required, online databases such as Northampton Electronic Library Search ONline (NELSON) and Education Research Complete were employed. Search terms such as “mastery”, “variation”, “conceptual understanding”, “mathematical fluency”, “mathematics” and “primary mathematics” were used. Additionally, using the Trends in International Mathematics and Science Study (TIMSS), top performing countries were identified, and their approaches analysed. Further literature was obtained by searching key names that were mentioned in various literatures, for example: Helen Drury, Anne Watson and John Mason, and Gu Lingyuan, through the databases previously mentioned and Google Scholar. Advice for direction was also sought from university lecturers. Finally, whilst on placement in a school, a training course for mastery was attended to collate more information (see Appendices 2 and 3).

However, this secondary research project was conducted through analysis and discussion of qualitative research, by exploring variation as a part of a mastery approach to teaching primary mathematics. Synthesis of existing research, from a variety of sources, will form new interpretations and help explore variation examples. This information will then be summarised for presentation for the reader, whilst professionally developing myself through unique insights into the research area.
Chapter 4 – Defining Variation

4.1 What is Variation?

An important element of mathematical understanding is having the ability to make connections between different ideas and concepts (Barmby et al., 2010, p.46; Skemp, 1976, p.2). Variation is the practice, which makes these connections possible, by varying unessential elements of exercises and questions, pupils are enabled to explore, discover and make connections for themselves, through repetition of rich problem solving (Gu et al., 2004, p.317; Lai and Murray, 2012, p.4; Ginsburg et al., 2005, p.ix). Variation is able to do this as it is centered on carefully structured questions, which invite pupils to reflect on the effects of their actions, where the desire is that they will recognise key relationships (Simon and Tzur, 2004, p.94). Additionally, variation is exposing pupils to concepts both concretely and abstractly to identify patterns and relationships in a specifically structured format, therefore igniting connections within the initial concept, and linking the concept to other areas of mathematics and the wider world (Cotton, 2013, p.110; Haylock, 2010, p.25; Barmby et al., 2010, p.46). Differentiation is also enabled through variation questioning, as all pupils, regardless of mathematical ability, can follow the same curriculum content, at the same pace (Stripp, 2014). This approach encourages depth before breadth, an essential element of mastery, as rather than accelerating ‘more able’ pupils onto new content, they focus on developing a deeper understanding of the same content. Whilst ‘lower ability’ pupils are not left behind, but supported through intervention to maintain the pace with the rest of their class (Mathematics Mastery, 2015; Drury, 2014, p.27).

Schools in Singapore currently follow this structure of curriculum content delivery, and in both the TIMSS, 2011 (Trends in International Mathematics and Science Study), at grades 4 and 8, and in PISA, 2012 (Programme for International Student Assessment), Singapore topped the international mathematics rankings. Ginsburg et al. (2005, p.ix) discussed how it is
unreasonable to assume their success is inherited, therefore it must be based on factors in their educational system. Figure 3, previously discussed in chapter 2, also demonstrates Singapore’s emphasis on metacognition, connections, problem solving and confidence, all of which are developed through variation. With Singapore topping international mathematics rankings, a rationale for examining and implementing features of their educational system, which could benefit pupils in England, is proposed.

4.2 Procedural and Conceptual Variation

Gu et al. (2004, p.315) and Lai and Murray (2012, p.8) examine the different types of variation; procedural and conceptual. Procedural variation uses modified and specifically selected questions to formulate concepts for pupils stage by stage. Hence developing their understanding of one particular concept, and form connections within this concept and different ways it can be approached (Appendix 3; Gu et al., 2004, p.320). For example, pupils might be given the question: 14- □ < 6; and the pupils will be required to explore the many possibilities for □. Hence enabling pupils to discover patterns and relationships, within this question and others that are similar. Effective procedural variation is therefore dynamic, as when pupils...
move between one calculation and the next there are connections for them to discover and explore, therefore providing pupils with the opportunity to focus on relationships and not just the mathematical procedure.

In contrast, conceptual variation provides pupils with multiple perspectives and experiences of a particular mathematical concept (Gu et al., 2004, p.315; Lai and Murray, 2012, p.8). For example in figure 4, when exploring fractions different lessons could use resources such as bead strings, arrays and unarranged groups of items, for pupils to explore and help build the blocks of understanding. Consequently, developing their mastery of the concept of fractions, whilst making connections with how fractions, the resources, and the procedures used, links to other areas of mathematics. Conceptual variation is therefore the practice of presenting one mathematical concept in a variety of ways to gain a deeper understanding. However, it is an ideal to assume that either type of variation can be so easily distinguished. As many effective variation questions, may achieve elements of both conceptual and procedural, dependent on the needs and prior experiences of the pupil. Both types of variation will provide pupils with intelligent practice, as although repetition is essential, it is not mechanical repetition, but opportunities for practicing mathematical thinking, reasoning and justifying to transform unsolved problems into solved problems (Lai and Murray, 2012, p.4; Gu et al., 2004, p.322).
4.3 How Variations fits in the English Curriculum

Both the new curriculum and the Ofsted handbook emphasise the objective - that mathematics teaching needs to create deep understanding, rather than accelerating pupils onto new content (NCETM, 2014). As variation prioritises deep understanding, whilst developing all pupils at the same pace, it also promotes high expectations of all pupils. Stripp (2013) argues that high expectations of all pupils is a priority. As under our current structure in England, those pupils labeled as ‘low’ have their access to knowledge and understanding restricted, resulting in them becoming further behind. Whilst ‘high-achieving’ pupils can have their progress limited, as they can become unwilling to tackle challenges incase it affects their perception of being clever. The Department for Education and Employment (2000a, p.5) also highlighted the need for all pupils to spend the same time on curriculum content, but the needs of more able pupils would be better served by going deeper in the topic, as opposed to the current practice of being accelerated onto new content. This highlights the need for change has been long recognised for over 16 years, yet has not been executed. Thus by implementing a mastery approach, variation will meet this need as pupils can be adequately challenged for their ability, with all pupils remaining at the same pace.

Simon and Tzur (2004, p.92) argue that mathematics teachers need to keep up to date with their understanding of the learning process, and with new concepts such as variation. As researchers have found, through variation pupils are enabled to look for similarities and make connections, which results in strong progress (Watson and Mason, 2006, p.92 & p.96). Therefore, lesson planning needs to consider what pupils will see, hear and think, and how they might respond, in order for teachers to develop effective variation tasks.

The Department for Education and Employment (2000b, p.5) also emphasised the requirement for more investigatory approaches in the classroom, with opportunities to make comparisons and connections. Variation also meets this requirement through: repetition of essential
elements in similar questions; providing learning experiences with concrete and abstract operations; and opportunities to work on problems that consist of many different elements and techniques (Gu et al., 2004, p.312-315). Oftsed (2011, p.6) concurred, stating good practice is when pupils’ confidence, fluency and reasoning are nurtured through rich problem solving. The new national curriculum (2013, p.3) supports and promotes a change to a classroom environment focused around deep understanding. The three key aims of the mathematics programme of study are fluency, reasoning and problem solving, hence providing justification for a mastery approach in the classroom. Through variation all aims can be achieved, not in one individual lesson, but over time when focusing on depth of content, as opposed to breadth, as can be argued English classrooms do currently (Drury, 2014, p.59; Bobis, 2007, p.22; Watson and Mason, 2006, p.92).
Chapter 5 – What Makes Variation an Effective Teaching Approach?

5.1 Depth Before Breadth

As variation is a feature of the mastery approach, depth before breadth is the underlining theme, which ensures all pupils have the opportunity to achieve in mathematics (Drury, 2014, p.26; Lai and Murray 2012, p.6). Stripp (2014) discusses that the use of mathematical representations, which expose the structure of mathematics, is a fundamental reason as to why pupils, who are taught through a mastery approach, are able to make sense of concepts and achieve fluency. Particularly in the primary years, Cockcroft (1982, p.84) concurs that practical work needs to have structure, with a wide range of experiences and followed by discussion. All of which are components of the mastery approach, which helps pupils to become successful learners, and develop a deep understanding of mathematical concepts (Haylock, 2010, p.3). However, it can be argued that in current English primary classrooms, pupils are developing a rote learning mindset. As many pupils have been taught mathematics procedures and routines without the chance to make sense of them (Haylock, 2010, p.24; Mathematics Mastery, 2015). Yet through variation, pupils have the opportunity to develop a meaningful learning mindset, as they are exposed to real objects, images and models, before applying their new knowledge to the abstract concept (Lai and Murray, 2012, p.6; Mathematics Mastery, 2015; Haylock, 2010, p.25). By doing so, they are encouraged to think mathematically, which leads to conceptual development, as well as fluency and accuracy (Watson and Mason, 2006, p.97).

As previously mentioned, Cockcroft (1982, p.84) also highlighted the requirement for reflective discussion. Discussion is another element of depth before breadth, as when pupils have the opportunity to use mathematical vocabulary and justify their reasoning, it supports the deepening of their learning (Thompson, 1999, p.4; Hansen, 2008, p.7). Discussion also enables pupils to make sense of what they are doing, as opposed to just learning and repeating the procedures (Haylock, 2010, p.3).
Additionally, by pupils discussing their strategies, they expose their peers to new perspectives and thinking, which can increase proficiency and understanding (Thompson, 1999, p.4). Therefore, as variation has a distinct ability to foster depth before breadth, it is one reason why it is an effective element of mastery.

5.2 Quality of Teaching

In the context of depth before breadth, variation will not be as successful when partnered with teachers who have little understanding or confidence in mathematics (Haylock, 2010, p.3). It is essential that teachers of the mastery approach, including variation, have the ability to emphasise the understanding of concepts. Yet, whilst this is common practice in many Eastern educational systems, through variation, in England teachers still appear to pay greater attention to procedural knowledge (Gu et al., 2004, p.311). However, when teachers have high standards, use a variety of strategies and are supported by their school to professionally develop, Ofsted (2008, p.7) found mathematics teaching and learning to be outstanding, as mathematical fluency was successfully being developed. In relation, although many schools do have highly qualified teachers, of which some are already following a variation approach, there remains to be schools who often place their ‘at-risk’ pupils with teaching assistants, who hold significantly varying levels of understanding, and are unable to apply a deepening approach (Ginsburg et al., 2005, p.ix).

Fundamentally, variation is a strong element of mastery, and when paired with confident and well-trained teachers, is a vital tool to deepening pupils’ mathematical understanding (Drury, 2014, p.25; Simon and Tzur, 2004, p.92). Appendix 2 puts into context how teachers with deep subject knowledge are able to: naturally use rich vocabulary to stimulate pupils’ interest and understanding; use this vocabulary to develop precise questioning which encourages pupils to think deeply and develop fluency; create and develop their own variation exercises to help pupils make
connections and recognise relationships; and identify resources and models which can concretely and abstractly put concepts into context.

Teachers with deep subject knowledge, who continue to remain up to date with training and new concepts, are able to plan and create effective variation tasks (Watson and Mason, 2006, p.91; Simon and Tzur, 2004, p.92). Teachers need to ensure lesson planning considers different sensory elements for the pupil and how they may perceive tasks. By doing so, carefully considered exercises are created, as by gradually introducing and repeating interconnected learning experiences, conceptual development and progress in all pupils is built, and motivation and interest can be fostered (Watson and Mason, 2006, p.97). As a consequence, the effectiveness of variation increases with the quality of teaching, as deep understanding will underpin the mathematics curriculum content and stimulate procedural and conceptual fluency in all pupils.

5.3 High Expectations of all Pupils

As variation enables all pupils to work through the curriculum content at the same pace, teachers are also enabled to have and display high expectations of all pupils (Drury, 2014, p.54; Mathematics Mastery). The mastery approach tackles the issue of too many pupils falling behind, and not enough exceeding, as every child is given the opportunity to work through the same tasks, listen and contribute in mathematically rich discussions, and explore mathematical problems individually and with their peers (Mathematics Mastery, 2015; Drury, 2014, p.48). Considerations of this kind are supported by Stripp (2014), who argues the mastery approach helps prevent some pupils missing out on curriculum content, and stops extension work creating the idea that success in mathematics is a race. Variation also provides differentiation through emphasising deep knowledge, and the recognition of connections between and within concepts. By doing so, the journey towards fluency and efficiency for all pupils is the main aim (Appendix 2). Variation can additionally be a form of differentiation, where pupils are given differing resources or time to tackle
the challenges and reason about them. Therefore, all pupils can continue to work through the same content, and not have their entire mathematical ability generically labeled, preventing their access to some content learning (Stripp, 2014; Mathematics Mastery, 2015; Ofsted, 2008, p.6). Every pupil is entitled to access the same content as their peers, therefore by keeping all abilities together, this enables pupils to be encouraged and inspired by one another, whilst having the opportunity to increase depth of knowledge through exploration and applying concepts through problem solving (Drury, 2014, p.24, p.27; Mathematics Mastery). Variation continues to be an effective element of the mastery curriculum, by helping to achieve the aim of high expectations of all pupils. As through these expectations, all learners are supported and enabled, previously labeled ‘low’ pupils are able to keep up and follow the same content as their peers. Whilst additionally, ‘high’ pupils are encouraged to make connections to deepen their understanding of the content.
Chapter 6 – Analysing Examples of Variation

6.1 Example 1 (Appendix 2)

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<td>= 16</td>
<td></td>
</tr>
<tr>
<td>18 -</td>
<td>= 10</td>
<td></td>
</tr>
<tr>
<td>16 -</td>
<td>= 12</td>
<td></td>
</tr>
<tr>
<td>14 -</td>
<td>= 6</td>
<td></td>
</tr>
<tr>
<td>12 -</td>
<td>= 8</td>
<td></td>
</tr>
</tbody>
</table>

This first example of variation could be argued as procedural variation. The procedure of finding an unknown is being explored, however the careful and specific choice of numbers chosen, helps pupils to identify patterns. The aim of the first column of questions is that after solving the initial problems, the pupil will then be able to identify a pattern. By doing so, they will no longer have to follow the procedure, but be able to apply the pattern to solve the remaining questions. Hence using one problem, to solve further problems. Making such connections deepens the pupil’s knowledge of the procedure and number facts (Appendix 2).

Subsequently, the second column of questions also enables the pupil to explore the procedure of finding the unknown. Yet, unlike the first column, there is no consistent number that all of the questions have in common. However, similarly to the first column, the questions continue to use even numbers, have the subtraction operation, and the unknown is in the same place. Therefore, pupils can explore patterns and relationships again, but this time deepening their understanding of both finding an unknown, and number facts, as increasing sophistication is used as the questions progress. Hence enabling teachers to use this example of variation to develop pupils’ understanding of a procedure, and a form of differentiation (Cotton, 2013, p.20; Department for Education, 2013, p.3).
6.2 Example 2  (Singapore, Maths Publishers, 2015, p.3 & p.43)

Problem 1: John has 7 balls. Mark has 5 more than John. How many balls does Mark have?

<table>
<thead>
<tr>
<th>John</th>
<th>7</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mark</td>
<td>7</td>
</tr>
</tbody>
</table>

Problem 2: How much does a toy bike cost if...

\[ \begin{align*}
\text{Cost of a toy car} &= £58 - £48 = £10 \\
\text{Cost of 2 toy bikes} &= £48 - £10 - £10 = £28 \\
\text{Cost of 1 toy bike} &= £28 \div 2 = £14
\end{align*} \]

The Singapore Bar Model is a form of variation accredited to Singapore. It is developed with pupils from the beginning of their education, as they learn to transfer physical objects, and then questions, to be represented by the abstract concept of the bar (Singapore, Maths Publishers, 2015, p.1). The Singapore Bar Model is an example of both procedural and conceptual
variation. Procedurally, the concept of the bar is being developed and deepened through further complex questions. Conceptually, it presents a mathematical problem in a number of ways, so the pupil can gain a deep understanding through exploring the question in a concrete and abstract form. In the first problem, a key stage one example, the bar is drawn to create a visual image, by doing so it makes it clear to the pupil how the numerical question has been developed from the worded question. It is essential when using the bar model to use and draw blocks which are proportionately representative, in order to avoid misunderstandings and misconceptions. By seeing the bar models side by side, the pupil can grasp the concept that the word ‘more’ in this question, symbolises addition.

The bar model can be used to represent a variety of mathematical problems. By introducing the bar model to pupils at a young age, they are then confident in using and applying the model to make more complex problems into simple problems. In the second problem, a key stage two example, the bar model is being used to represent a worded question involving money. By using the bar model, the pupil is enabled to find relationships between the total cost and cost of the individual items. By using the model, the pupil is continuing to be prepared to tackle further more complex questions, where they will be able to justify their answers and have a deep understanding of how they came to their answer (Singapore Maths Publishers, 2015, p.1; Mathematics Mastery, 2015).

6.3 Example 3 (Foong et al., 2015, p.60)

*Look at these shapes.*

![Shapes](image)

*How can we sort them?*
Amira sorted the shapes by number of sides.

<table>
<thead>
<tr>
<th>3 sides</th>
<th>4 sides</th>
<th>6 sides</th>
</tr>
</thead>
<tbody>
<tr>
<td>![3-sided shapes]</td>
<td>![4-sided shapes]</td>
<td>![6-sided shapes]</td>
</tr>
</tbody>
</table>

Can you help Ravi sort the shapes by the number of vertices?

<table>
<thead>
<tr>
<th>3 vertices</th>
<th>more than 3 vertices</th>
</tr>
</thead>
<tbody>
<tr>
<td>![3-vertex shapes]</td>
<td>![more than 3-vertex shapes]</td>
</tr>
</tbody>
</table>

Emma sorted the shapes in this way.

How did she do it?

This third example of variation is also an example of both procedural and conceptual variation. Procedurally, the concept of sorting is being developed, as well as the procedure of identifying properties of shapes. Furthermore, conceptually, the concepts of properties of shape are explored.
from different focuses to extend understanding. By initially presenting the pupils with a completed table sort, the pupils are immediately presented with geometry facts about the shapes they are going to be working with. This therefore can ignite discussion and confidence in pupils to tackle the subsequent questions. By already knowing the number of sides a shape has, every pupil is enabled to join in discussion and explore the relationship between what they know, and the task they are trying to complete.

The final table is already sorted for the pupils, hence further developing their conceptual understanding of 2D geometry. In this question the pupils are presented with the reverse perspective, of having to identify the properties the shapes have been grouped by. In the same way as other variation questions, this question enables pupils to reason and strengthen their mathematical understanding (Barmby et al., 2010, p.50). By using this example of variation successfully, variation can also be used as a form of assessment, as teachers can discern the level of potential distance between the pupil’s current level of knowledge, and the level of knowledge desired (Gu et al., 2004, p.326).

6.4 Example 4  (Gu et al., 2004, p.316)

Visual Model          Variation Figure

This example of variation is used to help pupils generalise their perceptions, and enable them to make connections between the concrete experience and the abstract mathematical concept (Gu et al., 2004, p.316). By doing so, the example is enabling pupils to deepen their understanding of the concept
from multiple perspectives, whilst additionally, developing their understanding and accuracy when applying the procedures needed to solve problems. These examples can then be broken down further, so pupils can apply the abstract mathematics. For example, the standard figures are often used by teachers to represent the abstract concept, however when exploring the concept in mathematically rich problems, the concept will not always appear in the standard form. Therefore, pupils need be able to identify concepts in a non-standard form (see example below).

<table>
<thead>
<tr>
<th>Standard Figure</th>
<th>Non-Standard Figure</th>
</tr>
</thead>
<tbody>
<tr>
<td>Perpendicular</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
</tr>
<tr>
<td>Parallelogram</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
</tr>
<tr>
<td>Height of a Triangle</td>
<td></td>
</tr>
</tbody>
</table>

It is essential to use variations of both the concrete and the abstract to ensure pupils have the opportunity to master the concept as far as their ability enables them, as opposed to being restricted by decreased content (Gu et al., 2004, p.317; Stripp, 2014). Teaching through variation enables deep understanding and potential mastery, as variation illustrates essential features of mathematical concepts, whilst demonstrating them in a variety of forms, so pupils can find and develop connections for themselves (Gu et al., 2004, p.315; Mathematics Mastery, 2015; Drury, 2014, p.39).
6.5 Example 5  (Watson and Mason, 2006, p.104)

Reduce to simplest terms:

(a) \( \frac{4}{12} \)  (b) \( \frac{36}{12} \)  (c) \( \frac{240}{300} \)  (d) 5:5  (e) \( ab:ab \)  (g) 2\( \frac{1}{4} \):1  (h) \( \frac{6ab}{3b} \)

This final example, is not one of variation, but of a standard set of exercise questions in which pupils may be asked to explore when learning about ratio. The specific instruction is to express each question in its simplest form. However, as there is no step-by-step change, or carefully constructed connection between each question, pupils simply have to find common terms in order to cancel. Therefore, the exercise can be fully completed with no engagement with the concept of ratio (Watson and Mason, 2006, p.105). As each question is different enough to require new thinking, there will be no increase in speed or reduction in effort, hence not developing mathematical fluency in the pupil. Similarly, as there are no gradual changes to be considered, the pupils are unable to identify relationships and patterns, hence not developing conceptual understanding (Hansen, 2008, p.6).

On reflection, this chapter highlights the importance of variation in all areas of mathematics. As by developing carefully constructed exercises, pupils are provided with the tools they need to develop the skills in order to become fluent mathematicians. They are enabled to predict, develop and identify relationships and patterns to build conceptual understanding and fluency. Whereas, unstructured exercises lead to rote learning, denying pupils of the essence and satisfaction that mathematics provides when making sense of the world around us (Watson and Mason, 2006, p.107; Hansen, 2008, p.6; Haylock, 2010, p.15).
Chapter 7 – Conclusion

As a result of the literature explored in this study, the effectiveness of mastery, with particular reference to variation, has become evident. The exploration into how variation is applied, and where it would fit into the national curriculum, has provided a rationale for its implementation into primary mathematics classrooms.

With reference to the literature analysed in chapter 4, the mastery approach including the element of variation is successfully being used in many countries, which are producing mathematically fluent pupils. Variation builds understanding of both concrete and abstract concepts, by gradually building the mental blocks between them, and using repetition to revisit between both concepts to secure understanding. Whilst there are two types of variation, procedural and conceptual, they cannot be easily distinguished in examples (Gu et al., 2004, p.315; Lai and Murray, 2012, p.8). Procedural variation is used to develop pupils’ understanding of a particular procedure through connections made between questions. Whereas conceptual variation, presents a mathematical concept in a variety of ways to deepen understanding. This intelligent practice has the ability to meet the three key aims of the mathematics programme of study: fluency, reasoning and problem solving (Department for Education, 2013, p.3).

The effectiveness of variation is however determined by execution. As explored in chapter 5, a teachers subject knowledge and confidence can restrict the benefits of variation (Haylock, 2010, p.3; Ginsburg et al., 2005, p.ix). As if they are unable to use precise and accurate mathematical vocabulary, they will be unable to encourage mathematical discussions and reasoning. Similarly, teachers who lack subject knowledge will also be unable to generate mathematically rich variation questions, preventing pupils from being able to build the conceptual blocks to understanding. As a result, pupils will be unable to deepen their knowledge of a concept, or take ownership of their learning to find patterns and connections. Variation is also a tool to help support teachers to have high expectations of all pupils (Drury, 2014, p.54; Stripp, 2014). When used as part of the mastery
approach, pupils will be kept together on the same content, and differentiated by having the opportunity to deepen their knowledge through variation tasks, or supported with rapid intervention.

Finally, chapter 6 analysed variation examples to identify the benefits of variation. As previously mentioned, distinguishing the examples as either procedural or conceptual was implausible, as most examples were able to develop pupils both procedurally and conceptually, a benefit of effective variation tasks. Moreover, variation was also a tool for differentiation, assessment, and igniting discussion to promote reasoning (Cotton, 2013, p.20; NCETM, 2014). Additionally, variation provides pupils with the building blocks to make connections independently. By doing so, pupils have the beauty of discovering mathematical patterns and connections for themselves, which can build mathematical confidence.

Through this research I have developed conclusions about effective variation. When variation is used as part of a mastery approach to teaching mathematics, it meets the key aims of mastery. Promoting deepening as a differentiation tool keeps all pupils at the same content pace, hence reinforcing the expectations that all pupils can succeed in mathematics. And variation can be the basis of precise questioning and development of deep conceptual and procedural knowledge to build fluency. However, it can be concluded that effective variation needs to be developed and executed by teachers with confidence and strong subject knowledge.

The importance of this research centers on the need to improve mathematics teaching and learning, hence ensuring all pupils are given the tools needed to achieve, and maintain a successful economy. By exploring an approach that has been effective for many mathematically high performing countries, this research has been able to analyse how the approach meets the needs of the national curriculum, whilst on balance, meeting the needs of pupils. As this research project used qualitative secondary research, future research could be conducted using long-term case studies. Using this literature study as a foundation, future research projects could examine the effects in pupil progress over time when
exposed to variation. For example, a study into the progress of pupils taught using variation, both in a mastery approach and on its own could be examined. On the other hand, the effectiveness of variation exercises created by teachers with varying subject knowledge could also be the focus of a subsequent study.

This research project however, does not under estimate the process and knowledge required to develop effective variation questions across all lessons, in all topics, in all year groups. As being able to identify the intermediate steps between concepts and experiences, is a large challenge in itself. Therefore, it is recommended that the need for collaboration and professional development is a key element in making variation, and mastery, successful long term.

With regard to myself professionally, this research project has enabled me to further understand the importance of orchestrating mathematical tasks. For instance, though a set of questions may appear to challenge the pupil in different ways, or ask questions using different formats, the depth of understanding and development in the pupil comes when the questions are carefully and specifically chosen. Furthermore, my interest in both mastery and variation has grown throughout conducting this project. For this reason, in time, I would like to continue research around both elements in further depth, through literature and conducting research, not only to advance my own understanding, but to ensure I offer the best possible mathematics education I can to all the children I teach.

On reflection, this research project has explored the issues and debates around changing the approach we have to teaching mathematics. In doing so, it has ascertained what high performing countries are currently doing, discussed the mastery approach, and reviewed how this would meet the current national curriculum. With particular focus on literature based on variation, this dissertation has highlighted the benefits of how the approach will build conceptual and procedural understanding in pupils, resulting in more pupils leaving school mathematically fluent.
References


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Appendix 2 – Calculation Guidance for Primary Schools

Calculation Guidance for Primary Schools.

One of the first projects undertaken by the Maths Hubs programme, launched in summer 2014, was the England-China school exchange, involving primary teachers in the school year 2014-15, and secondary teachers in the current school year, 2015-16. In both these exchanges, English teachers visit Shanghai for a fortnight and are immersed in a number of schools there, and then their exchange partner teachers from those schools come to England for a month and teach maths in the English teachers’ schools alongside their partners.

On 23 and 24 June 2015, the 71 teachers from the 47 lead primary schools (the LPS) who participated in the first year of the exchange gathered at the National College for Teaching and Leadership; together, they had hosted 55 Shanghai exchange teachers in their schools. They brought to Nottingham and shared with each other what they had learned from the exchange so far, and they also set out their plans for embedding and extending their knowledge in the coming academic year; in their own and also in their local partner schools.

One session of the conference focused in detail on “calculation guidance”; the purpose was to produce a list of recommendations and effective practice teaching ideas developed during and after the exchange visits. This document summarises and synthesises the discussions that took place and the reflections that were shared by the LPS teachers, and also draws on the two written reports submitted by each LPS during 2014-15. This is not intended to be a calculation policy as such; rather, it could sit alongside a school’s existing policy, and the ideas captured here (which are indicative and not exhaustive) could inform and enhance teaching across all primary key stages.

The following priority areas have been identified. Each of these is discussed in more detail with examples below:

- Develop children’s fluency with basic number facts.
- Develop children’s fluency in mental calculation.
Develop children’s fluency with basic number facts

Fluent computational skills are dependent on accurate and rapid recall of basic number bonds to 20 and times-tables facts. The LPS schools have found that spending a short time everyday on these basic facts quickly leads to improved fluency. This can be done using simple whole class chorus chanting. The LPS teachers are clear that this is not meaningless rote learning; rather, this is an important step to developing conceptual understanding through identifying patterns and relationships between the tables (for example, that the products in the 6\times table are double the products in the 3\times table). They have found that this has helped children develop a strong sense of number relationships, an important prerequisite for procedural fluency.

Children in Shanghai learn their multiplication tables in this order to provide opportunities to make connections:

- Develop children’s fluency in the use of written methods
- Develop children’s understanding of the = symbol
- Teach inequality alongside teaching equality
- Don’t count, calculate
- Look for pattern and make connections
- Use intelligent practice
- Use empty box problems
- Expose mathematical structure and work systematically
- Move between the concrete and the abstract
- Contextualise the mathematics
- Use questioning to develop mathematical reasoning
- Expect children to use correct mathematical terminology and speak in full sentences
- Identify difficult points
Develop children's fluency in mental calculation

Efficiency in calculation requires having a variety of mental strategies. In particular, the Shanghai teachers who participated in the exchange emphasised the importance of '10 and partitioning numbers to bridge through 10'. For example:

\[ 9 + 6 = 9 + 1 + 5 = 10 + 5 = 15. \]

The Shanghai teachers referred to "magic 10". It is helpful to make a 10 as this makes the calculation easier.

Develop fluency in the use of formal written methods

Teaching column methods for calculation provides the opportunity to develop both procedural and conceptual fluency. The LPS teachers noted that the Shanghai teachers ensured that children understood the structure of the mathematics presented in the algorithms, with a particular emphasis on place value. They saw base ten apparatus being used and illustrated in textbooks to support the development of fluency and understanding.

Informal methods of recording calculations are an important stage to help children develop fluency with formal methods of recording. A noticeable difference, however, that the LPS teachers observed in Shanghai is that these were only used for a short period, to help children understand the internal logic of formal methods of recording calculations. They are stepping stones to formal written methods. Here is an example from a Shanghai textbook:
Develop children's understanding of the \( = \) symbol

The symbol \( = \) is an assertion of equivalence. If we write:

\[
3 + 4 = 6 + 1
\]

then we are saying that what is on the left of the \( = \) symbol is necessarily equivalent to what is on the right of the symbol. But many children interpret \( = \) as being simply an instruction to evaluate a calculation, as a result of always seeing it used thus:

\[
3 + 4 = \\
5 \times 7 = \\
16 - 9 =
\]

If children only think of \( = \) as meaning "work out the answer to this calculation" then they are likely to get confused by empty box questions such as:
3 + □ = 8

Later they are very likely to struggle with even simple algebraic equations, such as:

3y = 18

One way to model equivalence such as \(2 + 3 = 5\) is to use balance scales.

Chinese textbooks vary the position of the equals symbol and include empty box problems from Grade 1 (equivalent to Year 2 in England) to deepen children's understanding of the equals symbol.

Teach inequality alongside teaching equality

To help young children develop their understanding of equality, they also need to develop understanding of inequality. Some of the LPS teachers have experimented with teaching inequality before, or at the same time as, equality (as they observed in lessons in Shanghai). One way to introduce the < and > signs is to use rods and cubes to make a concrete and visual representations such as:

![Diagram of inequality symbols](image)

to show that 5 is greater than 2 (5 > 2), 5 is equal to 5 (5 = 5), and 2 is less than 5 (2 < 5).

Balance scales can also be used to represent inequality.
Incorporating both equality and inequality into examples and exercises can help children develop their conceptual understanding. For example, in this empty box problem children have to decide whether the missing symbol is <, = or >:

\[ 5 + 7 \square 5 + 6 \]

An activity like this also encourages children to develop their mathematical reasoning: "I know that 7 is greater than 6, so 5 plus 7 must be greater than 5 plus 6."

Asking children to decide if number sentences are true or false also helps develop mathematical reasoning. For example, in discussing this statement:

\[ 4 + 6 + 8 > 3 + 7 + 9 \]

a child might reason that "4 plus 6 and 3 plus 7 are both 10. But 8 is less than 9. Therefore 4 + 6 + 8 must be less than 3 + 7 + 9, not more than 3 + 7 + 9."

In both these examples the numbers have been deliberately chosen to allow the children to establish the answer without actually needing to do the computation. This emphasises further the importance of mathematical reasoning.

Don’t count, calculate

Young children benefit from being helped at an early stage to start calculating, rather than relying on ‘counting on’ as a way of calculating. For example, with a sum such as:

\[ 4 + 7 = \]

Rather than starting at 4 and counting on 7, children could use their knowledge and bridge to 10 to deduce that because 4 + 6 = 10, so 4 + 7 must equal 11.
Look for pattern and make connections

The Shanghai teachers used a great many visual representations of the mathematics and some concrete resources. Understanding, however, does not happen automatically; children need to reason by and with themselves and make their own connections. The Shanghai teachers talked about getting children into good habits from Year 1 in terms of reasoning and looking for pattern and connections in the mathematics. The question "What's the same, what's different?" is used frequently to make comparisons. For example "What's the same, what's different between the three times table and the six times table?"

Use intelligent practice

Chinese children do engage in a significant amount of practice of mathematics through class- and homework exercises. However, in designing these exercises, the teacher is advised to avoid mechanical repetition and to create an appropriate path for practising the thinking process with increasing creativity (Gu, 1993). The practice that Chinese children engage in provides the opportunity to develop both procedural and conceptual fluency. Children are required to reason and make connections between calculations. The connections made improve their fluency.

For example:

| 2 × 3 = | 6 × 7 = | 9 × 8 = |
| 2 × 30 = | 6 × 70 = | 9 × 80 = |
| 2 × 300 = | 6 × 700 = | 9 × 800 = |
| 20 × 3 = | 60 × 7 = | 90 × 8 = |
| 200 × 3 = | 600 × 7 = | 900 × 8 = |

Shanghai Textbook Grade 2 (aged 7/8)
Use empty box problems

Empty box problems are a powerful way to help children develop a strong sense of number through intelligent practice. They provide the opportunity for reasoning and finding easy ways to calculate. They enable children to practise procedures, whilst at the same time thinking about conceptual connections.

A sequence of examples such as

\[
\begin{align*}
3 + \square &= 8 \\
3 + \square &= 9 \\
3 + \square &= 10 \\
3 + \square &= 11
\end{align*}
\]

helps children develop their understanding that the = symbol is an assertion of equivalence, and invites children to spot the pattern and use this to work out the answers.

This sequence of examples does the same at a deeper level:

\[
\begin{align*}
3 \times \square + 2 &= 20 \\
3 \times \square + 2 &= 23 \\
3 \times \square + 2 &= 26 \\
3 \times \square + 2 &= 29 \\
3 \times \square + 2 &= 35
\end{align*}
\]

Children should also be given examples where the empty box represents the operation, for example

\[
\begin{align*}
4 \times 5 &= 10 \square 10 \\
6 \square 5 &= 15 + 15
\end{align*}
\]
6 $\square$ 5 = 20 $\square$ 10
8 $\square$ 5 = 20 $\square$ 20
8 $\square$ 5 = 60 $\square$ 20

These examples also illustrate the careful use of variation to help children develop both procedural and conceptual fluency.

**Expose mathematical structure and work systematically**

Developing instant recall alongside conceptual understanding of number bonds to 10 is important. This can be supported through the use of images such as the example illustrated below:

![Shanghai Textbook Grade 1 (aged 6/7)](image)

The image lends itself to seeing pattern and working systematically and children can connect one number fact to another and be certain when they have found all the bonds to 5.

Using other structured models such as tens frames, part whole models or bar models can help children to reason about mathematical relationships.

Page 9 - Calculation Guidance NCETM October 2015
Connections between these models should be made, so that children understand the same mathematics is represented in different ways. Asking the question “What’s the same, what’s different?” has the potential for children to draw out the connections.

Illustrating that the same structure can be applied to any numbers helps children to generalise mathematical ideas and build from the simple to more complex numbers, recognising that the structure stays the same; it is only the numbers that change.

For example:
Move between the concrete and the abstract

Children's conceptual understanding and fluency is strengthened if they experience concrete, visual and abstract representations of a concept during a lesson. Moving between the concrete and the abstract helps children to connect abstract symbols with familiar contexts, thus providing the opportunity to make sense of, and develop fluency in the use of, abstract symbols.

For example, in a lesson about addition of fractions children could be asked to draw a picture to represent the sum $\frac{1}{4} + \frac{1}{8} = \frac{3}{8}$. Alternatively, or in a subsequent lesson, they could be asked to discuss which of three visual images correctly represents the sum, and to explain their reasoning:

![Visual Representations](image-url)
Contextualise the mathematics

A lesson about addition and subtraction could start with this contextual story:

"There are 11 people on a bus. At the next stop 4 people get on. At the next stop 6 people get off. How many are now on the bus?"

This helps children develop their understanding of the concepts of addition and subtraction. But during the lesson the teacher should keep returning to the story. For example, if the children are thinking about this calculation

$$14 - 8$$

then the teacher should ask the children:

"What does the 14 mean? What does the 8 mean?", expecting that children will answer:

"There were 14 people on the bus, and 8 is the number who got off."

Then asking the children to interpret the meaning of the terms in a sum such as $7 + 7 = 14$ will give a good assessment of the depth of their conceptual understanding and their ability to link the concrete and abstract representations of mathematics.

The four slides below are taken from a lesson delivered by one of the Shanghai teachers (Liu Dong)
Notice how each activity varies. The children are asked to:

Slide 1: Start with the story (concrete) and write the number sentence (abstract).

Slide 2: Start with the story (concrete) and complete it. Then write the number sentence (abstract).

Slide 3: Start with the number sentence (abstract) and complete the story (concrete).

Slide 4: Start with part of the story, complete two elements of it (concrete with challenge) and then write the number sentence (abstract).
The children move between the concrete and the abstract and back to the concrete,* with an increasing level of difficulty.

Use questioning to develop mathematical reasoning

Teachers' questions in mathematics lessons are often asked in order to find out whether children can give the right answer to a calculation or a problem. But in order to develop children's conceptual understanding and fluency there needs to be a strong and consistent focus on questioning that encourages and develops their mathematical reasoning.

This can be done simply by asking children to explain how they worked out a calculation or solved a problem, and to compare and contrast different methods that are described. The LPS teachers have found that children quickly come to expect that they need to explain and justify their mathematical reasoning, and they soon start to do so automatically — and enthusiastically. Some calculation strategies are more efficient and the LPS teachers noted that the Shanghai teachers scaffolded children's thinking to guide them to the most efficient methods, whilst at the same time valuing their own ideas.

Rich questioning strategies include:

- "What's the same, what's different?"

In this sequence of expressions, what stays the same each time and what's different?

<p>| | | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>23 + 10</td>
<td>23 + 20</td>
<td>23 + 30</td>
<td>23 + 40</td>
</tr>
</tbody>
</table>

Discussion of the variation in these examples can help children to identify the relationship between the calculations and hence to use the pattern to calculate the answers.

Footnote: Calculation Guidance NCETM October 2016
• "Odd one out"

Which is the odd one out in this list of numbers: 24, 15, 16 and 22?

This encourages children to apply their existing conceptual understanding. Possible answers could be:

"15 is the odd one out because it's the only odd number in the list."

"16 is the odd one out because it's the only square number in the list."

"22 is the odd one out because it's the only number in the list with exactly four factors."

If children are asked to identify an 'odd one out' in this list of products:

\[ 24 \times 3 \quad 36 \times 4 \quad 13 \times 5 \quad 32 \times 2 \]

they might suggest:

"36 \times 4 is the only product whose answer is greater than 100."

"13 \times 5 is the only product whose answer is an odd number."

• "Here's the answer: What could the question have been?"

Children are asked to suggest possible questions that have a given answer. For example, in a lesson about addition of fractions, children could be asked to suggest possible ways to complete this sum:

\[ \square + \square = \frac{3}{4} \]

• Identify the correct question
Here children are required to select the correct question:

A 3.5m plank of wood weighs 4.2 kg

The calculation was:

\[ 3.5 \div 4.2 \]

Was the question:

a. How heavy is 1m of wood?
b. How long is 1kg of wood?

* True or False

Children are given a series of equations are asked whether they are true or false:

\[
4 \times 6 = 23 \quad 4 \times 6 = 6 \times 4 \quad 12 \div 2 = 24 \div 4 \quad 12 \times 2 = 24 \times 4
\]

Children are expected to reason about the relationships within the calculations rather than calculate

* Greater than, less than or equal \(\geq\), \(\leq\), or \(\approx\)

\[
3.4 \times 1.2 \quad 3.4 \quad 5.76 \div 0.4 \quad 4.69 \times 0.1
\]

These types of questions are further examples of intelligent practice where conceptual understanding is developed alongside the development of procedural fluency. They also give pupils who are, to use Ofsted’s phrase, rapid graspers the opportunity to apply their understanding in more complex ways.
Expect children to use correct mathematical terminology and to express their reasoning in complete sentences.

The quality of children’s mathematical reasoning and conceptual understanding is significantly enhanced if they are consistently expected to use correct mathematical terminology (e.g. saying ‘digit’ rather than ‘number’) and to explain their thinking in complete sentences.

I say, you say, you say, you say, we all say.

This technique enables the teacher to provide a sentence stem for children to communicate their ideas with mathematical precision and clarity. These sentence structures often express key conceptual ideas or generalities and provide a framework to embed conceptual knowledge and build understanding. For example:

*If the rectangle is the whole, the shaded part is one third of the whole.*

Having modelled the sentence, the teacher then asks individual children to repeat this, before asking the whole class to chorus chant the sentence. This provides children with a valuable sentence for talking about fractions. Repeated use helps to embed key conceptual knowledge.

Another example is where children fill in the missing parts of a sentence; varying the parts but keeping the sentence stem the same. For example:
There are 12 stars, \( \frac{1}{3} \) of the stars is equal to 4 stars.

Children use the same sentence stem to express other relationships. For example:

There are 12 stars, \( \frac{1}{4} \) of the stars is equal to 3 stars.

There are 12 stars, \( \frac{1}{2} \) of the stars is equal to 6 stars.

Similarly:

There are 15 pears, \( \frac{1}{3} \) of the pears is equal to 5 pears.

There are 15 pears, \( \frac{1}{5} \) of the pears is equal to 3 pears.
When talking about fractions it is important to make reference to the whole and the part of the whole in the same sentence. The above examples help children to get into the habit of doing so.

Another example is where a mathematical generalisation or “rule” emerges within a lesson. For example:

*When adding 10 to a number, the ones digit stays the same*

This is repeated in chorus using the same sentence, which helps to embed the concept.

**Identify difficult points**

Difficult points need to be identified and anticipated when lessons are being designed and these need to be an explicit part of the teaching, rather than the teacher just responding to children’s difficulties if they happen to arise in the lesson.

The teacher should be actively seeking to uncover possible difficulties because if one child has a difficulty it is likely that others will have a similar difficulty. Difficult points also give an opportunity to reinforce that we learn most by working on and through ideas with which we are not fully secure or confident. Discussion about difficult points can be stimulated by asking children to share thoughts about their own examples when these show errors arising from insufficient understanding. For example:

\[
\frac{2}{14} - \frac{1}{7} = \frac{1}{7}
\]

A visualiser is a valuable resource since it allows the teacher quickly to share a child’s thinking with the whole class.
The New Curriculum

Factual & Procedural Fluency

Conceptual Understanding

INTEGRATION
The Knowledge Led Curriculum

3 Forms of Knowledge
Factual – I know that
Procedural – I know how
Conceptual – I know why
A Mastery Curriculum

- All/most pupils can and will achieve
- Keeping the class working together so that all can master mathematics
- Development of deep mathematical knowledge
- Development of both factual/procedural and conceptual fluency
- Longer time on key topics
What does it mean to master something?

- I know how to do it
- It becomes automatic and I don’t need to think about it - for example driving a car
- I’m really good at doing it – painting a room, or a picture
- I can show someone else how to do it.
Mastery of Mathematics is more.....

- Deep and sustainable learning
- The ability to build on something that has already been mastered
- The ability to reason about a concept and make connections
- Procedural fluency AND Conceptual fluency
Procedural Variation

Procedural variation is dynamic, where I move between one calculation and the next there is a connection. Children need to be taught from an early age to look for and recognise these connections.
Procedural Variation

Provides the opportunity
• To focus on relationships, not just the procedure
• To make connections between problems
• Using one problem to work out the next
Examples of Procedural Variation

7 + 2 = 9 + 6 = 18 - □ = 8
17 + 2 = 10 + 6 = 20 - □ = 16
7 + 12 = 11 + 6 = 18 - □ = 10
17 + 12 = 13 + 6 = 18 - □ = 12
9 - 5 = 9 - 7 = 18 - □ = 14
8 - 5 = 11 - 7 = 14 - □ = 6
7 - 5 = 13 - 7 = 18 - □ = 16
6 - 5 = 15 - 7 = 12 - □ = 8

9 × 50 □ 90 × 5

300 × 3 □ 5 × 200

907 - 100 = 807
907 - 99 = □
907 - 101 = □

888 - 99 = □
888 - 100 = □
888 - 101 = □

14 - □ < 6

75 + □□ = 100
56 + □□□ = 100

Singapore bar model.
Another and Another

Take a number ending in 7 and add 6
Repeat for another and another...........
What do you notice?

Answer the following
467 + 6 =
1,487 + 6 =
Conceptual Variation

The practice of presenting one maths concept in a number of different ways
Looking at the concept from multiple perspectives to gain deep understanding
Developing depth through representations

MathsHUBS

1. 先用分数表示下面各图中的涂色部分。

2. 涂色部分各是多少米？

3. 按所给的分数画一画。

4. 12个☆的$\frac{3}{4}$是几个☆？

Dead strings  arrays  unarranged

$\frac{3}{4}$  $\frac{4}{5}$  $\frac{5}{6}$
What is a fraction?

The relationship between a whole and the parts.

A whole can be divided into many parts.

Many parts can make one whole.

- Milton Keynes is the whole
  - Laughton is the part.
- England is the whole
  - Milton Keynes is the part ...
- The world is the whole
  - Europe is the part

→ Can you think of other
  - family is whole.
  - I am a part
Carefully chosen examples

(1)

(2)
A. Use fractions to express the coloured parts.
True or False
Conceptual and Non Conceptual Variation

\[
\begin{align*}
\frac{1}{2} \sqrt{} & \quad  \frac{1}{2} \times \quad  \frac{1}{3} \times \quad  \frac{1}{4} \times
\end{align*}
\]

Why, explain?
Teaching for Mastery

- **CONCEPTUAL VARIATION** to provide pupils with **multiple perspectives and experiences of mathematical concepts**.
- **PROCEDURAL VARIATION** to provide a process for **formation of concepts stage by stage**, in which pupils' experience in solving problems is manifested by the **richness of varying problems and the variety of transferring**.
- **INTELLIGENT PRACTICE**: when designing exercises, the teacher is advised to **avoid mechanical repetition and to create an appropriate path for practising the thinking process with increasing creativity**.

Gu, 1991